Exponential Functions

***Laws of Exponents*:** If s, t, a, and b are real numbers with $a > 0$ and $b > 0$,

then $a^s \cdot a^t = a^{s+t}$ $\quad (a^s)^t = a^{st}$ $\quad (ab)^s = a^s b^s$ $\quad 1^s = 1$ $\quad a^{-s} = \dfrac{1}{a^s} = \left(\dfrac{1}{a}\right)^s$ $\quad a^0 = 1$

An ***exponential function*** is a function of the form $f(x) = a^x$, where a is a positive real number $(a > 0)$ and $a \neq 1$. The domain of f is the set of all real numbers.

Theorem: For an exponential function $f(x) = a^x$, $a > 0$, $a \neq 1$, if x is any real number, then

$$\frac{f(x+1)}{f(x)} = a$$

Examples: Determine whether the given functions are exponential or not.

x	$f(x)$
-1	2
0	5
1	8
2	11
3	14

$5 \div 2 = \frac{5}{2}$

$8 \div 5 = \frac{8}{5}$

$11 \div 8 = \frac{11}{8}$

$14 \div 11 = \frac{14}{11}$

$\boxed{\text{not exponential}}$

x	$f(x)$
-1	2/3
0	1
1	3/2
2	9/4
3	27/8

$1 \div \frac{2}{3} = \frac{3}{2}$

$\frac{3}{2} \div 1 = \frac{3}{2}$

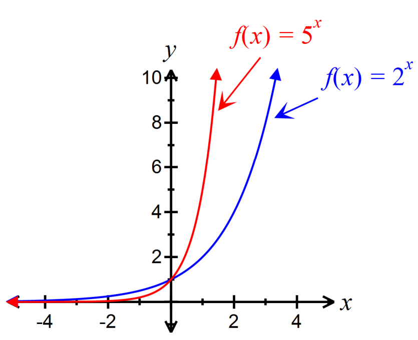
$\frac{9}{4} \div \frac{3}{2} = \frac{3}{2}$

$\frac{27}{8} \div \frac{9}{4} = \frac{3}{2}$

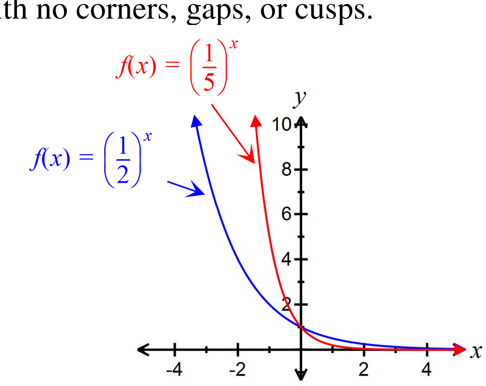
$\boxed{\text{exponential}}$

Properties of the Exponential Function $f(x) = a^x$, $a > 0$, $a \neq 1$

- Domain: $(-\infty, \infty)$; Range: $(0, \infty)$
- There are no x-intercepts; the y-intercept is 1.
- The x-axis $(y = 0)$ is a horizontal asymptote.
  - For $a > 1$, the graph approaches the x-axis as $x \to -\infty$.
  - For $0 < a < 1$, the graph approaches the x-axis as $x \to \infty$.
- $f(x) = a^x$ is one-to-one.
  - For $a > 1$, $f(x) = a^x$ is an increasing function.
  - For $0 < a < 1$, $f(x) = a^x$ is a decreasing function.
- The graph of f contains the points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$.
- The graph of f is smooth and continuous, with no corners, gaps, or cusps.



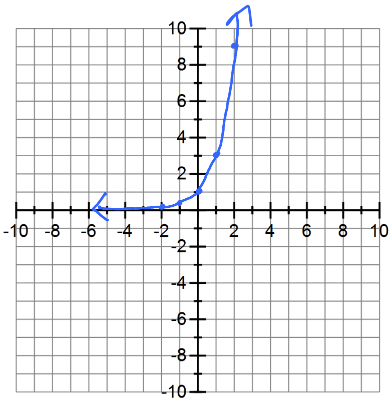
$a > 1$



$0 < a < 1$

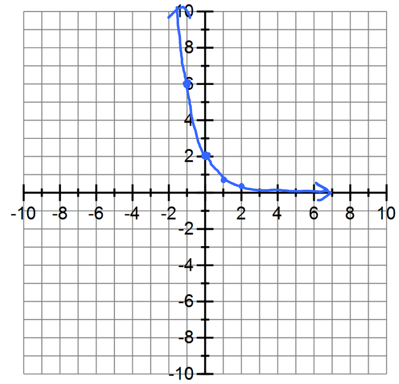
Examples:

a) Graph $f(x)=3^x$.



x	y
-2	1/9
-1	1/3
0	1
1	3
2	9

b) Graph $f(x)=2\cdot\left(\frac{1}{3}\right)^x$.

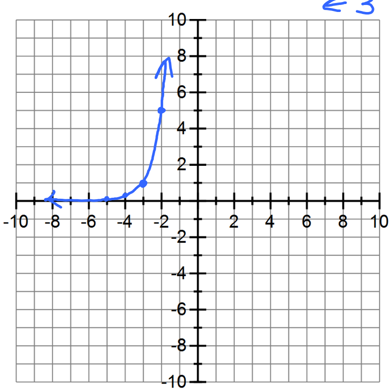


x	y
-2	18
-1	6
0	2
1	2/3
2	2/9

c) Graph $f(x)=5^{x+3}$.

parent: $y=5^x$

$\leftarrow 3$

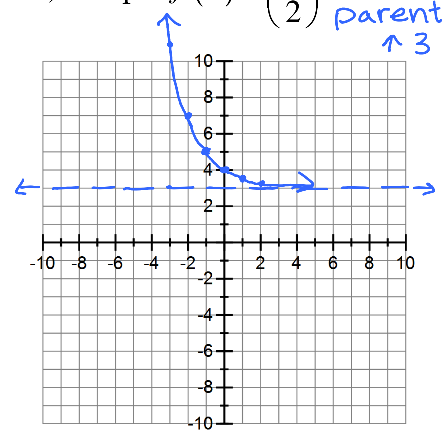


x	y
-5	1/25
-4	1/5
-3	1
-2	5
-1	25

d) Graph $f(x)=\left(\frac{1}{2}\right)^x+3$.

parent: $y=\left(\frac{1}{2}\right)^x$

$\uparrow 3$

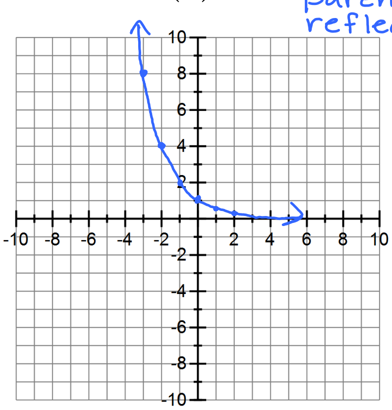


x	y
-2	7
-1	5
0	4
1	$3\frac{1}{2}$
2	$3\frac{1}{4}$

e) Graph $f(x)=2^{-x}$.

parent: $y=2^x$

reflect over y-axis

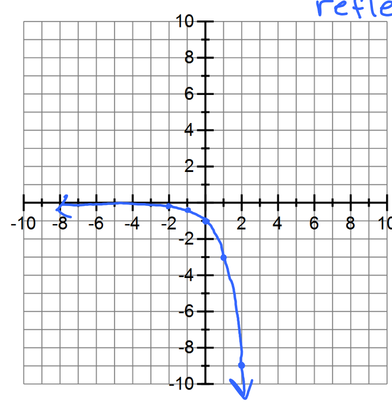


x	y
-3	8
-2	4
-1	2
0	1
1	1/2
2	1/4

f) Graph $f(x)=-3^x$

parent: $y=3^x$

reflect over x-axis

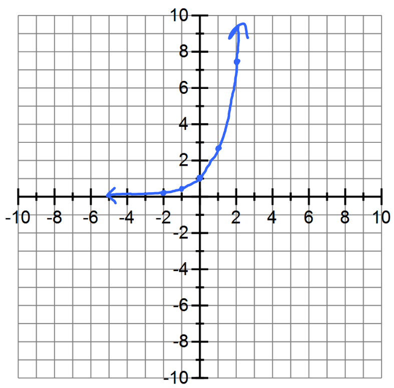


x	y
-2	-1/9
-1	-1/3
0	-1
1	-3
2	-9

The **number *e*** (approximately 2.71828…) is defined as the number that the expression $\left(1+\frac{1}{n}\right)^n$ approaches as $n \to \infty$. In calculus, this is expressed using limit notation as $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

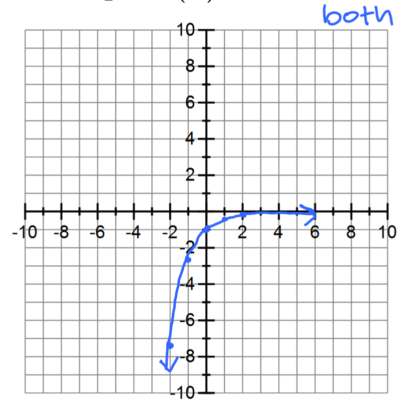
Examples:

a) Graph $f(x) = e^x$.



x	y
-2	.1
-1	.4
0	1
1	2.7
2	7.4

b) Graph $f(x) = -e^{-x}$ reflect $y=e^x$ over both x- & y-axis



x	y
-2	-7.4
-1	-2.7
0	-1
1	-.4
2	-.1

Solving Exponential Equations

If $a > 0$ and $a \neq 1$ and $a^u = a^v$, then $u = v$.

Many exponential equations can be rewritten so the two sides have a common base. This allows us to set the exponents equal to each other and solve the equation.

Examples: Solve the following equations.

a) $3^{-x} = 243$

$3^{-x} = 3^5$

$-x = 5$

$\boxed{x = -5}$

b) $5^{x+3} = \frac{1}{5}$

$5^{x+3} = 5^{-1}$

$x + 3 = -1$

$\boxed{x = -4}$

c) $4^{x^2} = 2^x$

$(2^2)^{x^2} = 2^x$

$2^{2x^2} = 2^x$

$2x^2 = x$

$2x^2 - x = 0$

$2x(x-1) = 0$

$\boxed{x = 0 \text{ or } x = 1}$

d) $3^{x^2-5x} = \frac{1}{81}$

$3^{x^2-5x} = 3^{-4}$

$x^2 - 5x = -4$

$x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$\boxed{x = 4 \text{ or } x = 1}$

e) $4^x \cdot 2^{x^2} = 16^2$

$(2^2)^x \cdot 2^{x^2} = (2^4)^2$

$2^{2x} \cdot 2^{x^2} = 2^8$

$2^{2x+x^2} = 2^8$

$2x + x^2 = 8$

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

$\boxed{x = -4 \text{ or } x = 2}$

f) $e^{x^2} = e^{3x} \cdot \frac{1}{e^2}$

$e^{x^2} = e^{3x} \cdot e^{-2}$

$e^{x^2} = e^{3x-2}$

$x^2 = 3x - 2$

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$\boxed{x = 2 \text{ or } x = 1}$